High-order well-balanced methods for the shallow-water model with moments

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Shallow water equations (SWE)

Shallow water equations (SWE)

$$\partial_t \begin{pmatrix} h \\ hu_m \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -gh\partial_x b \end{pmatrix} - \frac{\nu}{\lambda} \begin{pmatrix} 0 \\ u_m \end{pmatrix}, \quad (1)$$

where $u_m = u_m(t,x)$ is the horizontal water velocity, $h = h(t,x)$ is the water height, g is the gravitational constant, the known function $b(x)$ is the bottom topography, and ν and λ are the kinematic viscosity and the slip length, respectively.

Shallow water moment equations (SWME)

Derived in (Kowalski & Torrilhon, 2019), the idea is to allow for a vertical variation of the water velocity profile. This is done by assuming the following ansatz for the velocity profile:

$$u(t,x,\zeta) = u_m(t,x) + \sum_{j=1}^{N} \alpha_j(t,x)\phi_j(\zeta), \tag{2}$$

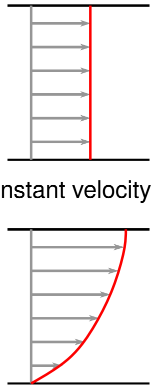
where $u_m(t,x)$ is the mean horizontal velocity also used in the SWE, ζ is the scaled vertical coordinate:

$$\zeta = \frac{z-b}{h}, \tag{3}$$

α_j are coefficients, and ϕ_j are Legendre ansatz functions for $j = 1, \dots, N$ defined by

$$\phi_j(\zeta) = \frac{1}{j!}\frac{d^j}{d\zeta^j}(\zeta - \zeta^2)^j. \tag{4}$$

Shallow water moment equations (SWME)



(a) Constant velocity profile

(b) Varying velocity profile

Figura: Constant velocity ansatz of SWE model (a) and variable velocity ansatz of SWME model (b).

Shallow water moment equations (SWME)

Shallow water moment equations (SWME) $N = 1$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 + \frac{1}{3}h\alpha_1^2 \\ 2hu_m\alpha_1 \end{pmatrix} = Q\partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \end{pmatrix} - \begin{pmatrix} 0 \\ gh\partial_x b \\ 0 \end{pmatrix} - \frac{\nu}{\lambda} P, \quad (5)$$

with

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u_m \end{pmatrix}, P = \begin{pmatrix} 0 \\ u_m + \alpha_1 \\ 3\left(u_m + \alpha_1 + 4\frac{\lambda}{h}\alpha_1\right) \end{pmatrix}.$$

Shallow water moment equations (SWME)

Shallow water moment equations (SWME) $N = 2$

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 + \frac{1}{5}h\alpha_2^2 \\ 2hu_m\alpha_1 + \frac{4}{5}h\alpha_1\alpha_2 \\ 2hu_m\alpha_2 + \frac{2}{3}h\alpha_1^2 + \frac{2}{7}h\alpha_2^2 \end{pmatrix} = Q\partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ h\alpha_2 \end{pmatrix} - \begin{pmatrix} 0 \\ gh\partial_x b \\ 0 \\ 0 \end{pmatrix} - \frac{\nu}{\lambda}P \tag{6}$$

with

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_m - \frac{\alpha_2}{5} & \frac{\alpha_1}{5} \\ 0 & 0 & \alpha_1 & u_m + \frac{\alpha_2}{7} \end{pmatrix} \text{ and } P = \begin{pmatrix} 0 \\ u_m + \alpha_1 + \alpha_2 \\ 3\left(u_m + \alpha_1 + \alpha_2 + 4\frac{\lambda}{h}\alpha_1\right) \\ 5\left(u_m + \alpha_1 + \alpha_2 + 12\frac{\lambda}{h}\alpha_2\right) \end{pmatrix}.$$

Shallow water linearized moment equations (SWLME)

$$\partial_t \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ \vdots \\ h\alpha_N \end{pmatrix} + \partial_x \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 + \ldots + \frac{1}{2N+1}h\alpha_N^2 \\ 2hu_m\alpha_1 \\ \vdots \\ 2hu_m\alpha_N \end{pmatrix} = Q\partial_x \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ \vdots \\ h\alpha_N \end{pmatrix}. \tag{7}$$

The non-conservative term is simplified to

$$Q = \mathrm{diag}\left(0, 0, u_m, \ldots, u_m\right).$$

Shallow water linearized moment equations (SWLME)

The system (7) with topography but without friction terms is therefore written in the form

$$U_t + \partial_x F(U) + B(U)\partial_x U = S(U)\partial_x b. \tag{8}$$

By straightforward calculation, we obtain

$$U = \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ \vdots \\ h\alpha_N \end{pmatrix}, \quad F(U) = \begin{pmatrix} hu_m \\ hu_m^2 + g\frac{h^2}{2} + \frac{1}{3}h\alpha_1^2 + \ldots + \frac{1}{2N+1}h\alpha_N^2 \\ 2hu_m\alpha_1 \\ \vdots \\ 2hu_m\alpha_N \end{pmatrix}, \tag{9}$$

$$B(U) = diag(0, 0, -u_m, ..., -u_m). \tag{10}$$

$$S(U) = (0, -gh, 0, ..., 0)^T. \tag{11}$$

Shallow water linearized moment equations (SWLME)

We can also write this system in the form

$$\partial_t W + \mathcal{A}(W)\partial_x W = 0, \tag{12}$$

with

$$W = \begin{pmatrix} h \\ hu_m \\ h\alpha_1 \\ \vdots \\ h\alpha_N \\ b \end{pmatrix}, \quad \mathcal{A}(W) = \begin{pmatrix} A(W) & -S(W) \\ 0 & 0 \end{pmatrix},$$

Shallow water linearized moment equations (SWLME)

where

$$A_N = \begin{pmatrix} 0 & 1 & 0 & \vdots & 0 \\ gh - u_m^2 - \frac{\alpha_1^2}{3} - \ldots - \frac{\alpha_N^2}{2N+1} & 2u_m & \frac{2\alpha_1}{3} & \ldots & \frac{2\alpha_N}{2N+1} \\ -2u_m\alpha_1 & 2\alpha_1 & u_m & & \\ \vdots & \vdots & & \ddots & \\ -2u_m\alpha_N & 2\alpha_N & & & u_m \end{pmatrix} \in \mathbb{R}^{(N+2)} \tag{13}$$

Eigenvalues of SWLME

Theorem (Koellermeier, Pimentel-García)

The SWLME system matrix $A_N \in \mathbb{R}^{(N+2)\times(N+2)}$ (13) *has the following characteristic polynomial*

$$\chi_{A_N}(\lambda) = (u_m - \lambda)\left[(\lambda - u_m)^2 - gh - \sum_{i=1}^{N} \frac{3\alpha_i^2}{2i+1}\right]$$

and the eigenvalues are given by

$$\lambda_{1,2} = u_m \pm \sqrt{gh + \sum_{i=1}^{N} \frac{3\alpha_i^2}{2i+1}} \quad \textit{and} \quad \lambda_{i+2} = u, \ \textit{for} \ i = 1, \ldots, N. \tag{14}$$

The system is thus hyperbolic.

Stationary solutions of SWLME

Looking for the stationary solutions: $\mathcal{A}(W)W_x = 0$, we can derive from the second equation

$$\partial_x \left(\frac{1}{2}u_m^2 + g(h+b) + \frac{3}{2}\sum_{i=1}^{N} \frac{1}{2i+1}\alpha_i^2 \right) = 0. \qquad (15)$$

The non-trivial steady state solution can thus be found using

$$\begin{aligned} hu_m &= const, \\ \frac{1}{2}u_m^2 + g(h+b) + \frac{3}{2}\sum_{i=1}^{N} \frac{1}{2i+1}\alpha_i^2 &= const, \\ \frac{\alpha_i}{h} &= const, \text{ for } i = 1, \dots, N. \end{aligned}$$

A high-order well-balanced methodology

Let us consider systems of the form

$$\partial_t U + \partial_x F(U) + B(U)\partial_x U = S(U)\partial_x b. \tag{16}$$

As we have seen these systems are equivalent to

$$\partial_t W + \mathcal{A}(W)\partial_x W = 0, \tag{17}$$

where

$$W = \begin{pmatrix} U \\ b \end{pmatrix}, \quad \mathcal{A}(W) = \begin{pmatrix} \frac{\partial F}{\partial U}(U) + B(U) & -S(U) \\ 0 & 0 \end{pmatrix}.$$

A high-order well-balanced methodology

We consider semi-discrete finite-volume methods of the form

$$\frac{dW_i}{dt} = -\frac{1}{\Delta x}\left(D^-_{i+\frac{1}{2}} + D^+_{i-\frac{1}{2}} + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathcal{A}(\mathbb{P}_i(x))\frac{\partial}{\partial x}\mathbb{P}_i(x)dx\right), \quad (18)$$

where

- $W_i(t) \cong \int_{x_{i+\frac{1}{2}}}^{x_{i-\frac{1}{2}}} W(t,x)\,dx$ is the cell average value,
- $\mathbb{P}_i(x)$ is the approximation of the solution at the *i*th cell given by a high-order reconstruction operator from the sequence of cell averages $\{W_i(x)\}$:

$$\mathbb{P}^t_i(x) = \mathbb{P}^t_i(x; \{W_j(t)\}_{j\in\mathcal{S}_i}), \quad (19)$$

where $\mathcal{S}_i$ denotes the set of indices of the cells belonging to the stencil of the *i*th cell.

A high-order well-balanced methodology

- $D^{\pm}_{i+\frac{1}{2}} = \mathbb{D}^{\pm}\left(W^{-}_{i+\frac{1}{2}}, W^{+}_{i+\frac{1}{2}}\right)$, is the respective fluctuation with reconstructed states

$$W^{-}_{i+\frac{1}{2}} = \mathbb{P}_i(x_{i+\frac{1}{2}}), \quad W^{+}_{i+\frac{1}{2}} = \mathbb{P}_{i+1}(x_{i+\frac{1}{2}}),$$

and $\mathbb{D}(W_l, W_r)$ verifies:

$$\mathbb{D}^{-}(W_l, W_r) + \mathbb{D}^{+}(W_l, W_r) = \int_0^1 \mathcal{A}(\Psi)\frac{\partial \Psi}{\partial s}\, ds, \qquad (20)$$

where Ψ is a family of paths joining W_l with W_r.

A high-order well-balanced methodology

The stationary solutions of the system (17) are those solutions of the system such that $W_t = 0$ and verify

$$\mathcal{A}(W(x))W_x(x) = 0, \tag{21}$$

for all x where W is defined.

- The numerical method (18) is said to be well-balanced for W^* if the vector of cell averages of W^* is an equilibrium of the ODE system (18).
- The reconstruction operator is said to be well-balanced for W^* if

$$\mathbb{P}_i(r) = W^*(r), \qquad r \in [r_{i-\frac{1}{2}}, r_{i+1/2}], \tag{22}$$

where $\mathbb{P}_i$ is the approximation of W^* obtained by applying the reconstruction operator to the vector of cell averages of W^*.

A high-order well-balanced methodology

Following (Castro,Gallardo,López-García & Parés, 2008) in order to compute a well-balanced reconstruction operator $\mathbb{P}_i$ at the cell $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ for a given family of cell values $\{W_i\}$:

1. Look for the stationary solution $W_i^*(x) = (U_i^*(x), \sigma(x))^T$ such that:
$$\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i-\frac{1}{2}}} W_i^*(x)dx = W_i, \tag{23}$$
if possible. In other cases consider $W_i^* \equiv W_i$.
2. Compute the fluctuations $\{V_j\}_{j\in S_i}$ within the stencil S_i:
$$V_j = W_j - \frac{1}{\Delta r}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} W_i^*(x)dx, \quad j \in S_i. \tag{24}$$

A high-order well-balanced methodology

3. Apply the standard reconstruction operator to the fluctuations $\{V_j\}_{j\in S_i}$:

$$\mathbb{Q}_i(x) = \mathbb{Q}_i(x; \{V_j\}_{j\in S_i}).$$

4. Define the well-balanced operator:

$$\mathbb{P}_i(x) = W_i^*(x) + \mathbb{Q}_i(x).$$

$\mathbb{P}_i$ is well-balanced for every steady solution provided that the reconstruction operator Q_i is exact for the null function. Moreover, it is conservative, i.e.,

$$\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbb{P}_i(x)dr = W_i, \quad \text{for all } i,$$

provided that Q_i is conservative, and it is high-order accurate provided that the steady solutions are smooth.

A high-order well-balanced methodology

In general the averages of the initial condition is computed using a cuadrature formula:

$$W_{0,i} = \sum_{k=0}^{M} \alpha_k^i W_0(x_k^i), \quad \forall i, \tag{25}$$

The two first steps are modified:

1. Look for the stationary solution $W_i^*(x)$ such that:

$$\sum_{k=0}^{M} \alpha_k^i W_i^*(x_k^i) = W_i. \tag{26}$$

2. Compute the fluctuations $\{V_j^n\}_{j\in S_i}$ within the stencil S_i:

$$V_j = W_j - \sum_{k=0}^{M} \alpha_k^j W_i^*(x_k^j), \;\; j \in S_i. \tag{27}$$

A high-order well-balanced methodology

We rewrite the semi-discrete (18) system as proposed in (Castro & Parés, 2020) taking into account the non-conservative part

$$\frac{dW_i}{dt} = -\frac{1}{\Delta x}\left(D^-_{i+\frac{1}{2}} + D^+_{i-\frac{1}{2}} + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\mathcal{A}(\mathbb{P}_i(x))\frac{\partial}{\partial x}\mathbb{P}_i(x) - \mathcal{A}(W_i^*(x))\frac{\partial}{\partial x}W_i^*(x)\right) dx. \right. \tag{28}$$

$$\begin{aligned}\frac{dW_i}{dt} = &-\frac{1}{\Delta x}\Big(D^-_{i+\frac{1}{2}} + D^+_{i-\frac{1}{2}} \\ &+ F(\mathbb{P}_i(x_{i+\frac{1}{2}})) - F(W_i^*(x_{i+\frac{1}{2}})) + F(W_i^*(x_{i-\frac{1}{2}})) - F(\mathbb{P}_i(x_{i-\frac{1}{2}})) \\ &+ \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(B(\mathbb{P}_i(x))\frac{\partial}{\partial x}\mathbb{P}_i(x) - B(W_i^*(x))\frac{\partial}{\partial x}W_i^*(x)\right) dx \\ &+ \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left((S(\mathbb{P}_i(x)) - S(W_i^*(x)))\frac{\partial}{\partial x}b(x)\right) dx.\end{aligned}$$

A high-order well-balanced methodology

Once this equivalent form is obtained, the quadrature formula can be applied to the integrals without losing the well-balanced property, and this leads to a numerical method of the form:

$$
\begin{aligned}
W_i'(t) = &-\frac{1}{\Delta x}\Bigg(D^-_{i+\frac{1}{2}} + D^+_{i-\frac{1}{2}} \\
&+ F(\mathbb{P}_i(x_{i+\frac{1}{2}})) - F(W_i^*(x_{i+\frac{1}{2}})) + F(W_i^*(x_{i-\frac{1}{2}})) - F(\mathbb{P}_i(x_{i-\frac{1}{2}})) \\
&+ \sum_{k=0}^{M} \alpha_k^i \left(B(\mathbb{P}_i(x_k^i))\frac{\partial}{\partial x}\mathbb{P}_i(x_k^i) - B(W_i^*(x_k^i))\frac{\partial}{\partial x}W_i^*(x_k^i) \right) \\
&+ \sum_{k=0}^{M} \alpha_k^i \left(S(\mathbb{P}_i(x_k^i)) - S(W_i^*(x_k^i))\frac{\partial}{\partial x}b(x_k^i) \right) \Bigg).
\end{aligned}
\tag{29}
$$

First-order well-balanced scheme for the SWLME

The cell averages of the initial condition will be computed using the mid-point rule, that is

$$W_i^0 = W_0(x_i), \quad \text{for all } i.$$

In the case of the SWLME system, the steady state solutions verify:

$$\begin{aligned}
hu_m &= C_1 \equiv const,\\
\frac{1}{2}u_m^2 + g(h+b) + \frac{3}{2}\sum_{i=1}^{N}\frac{1}{2i+1}\alpha_i^2 &= C_2 \equiv const,\\
\frac{\alpha_1}{h} &= C_3 \equiv const,\\
\frac{\alpha_2}{h} &= C_4 \equiv const,\\
&\vdots\\
\frac{\alpha_N}{h} &= C_{N+2} \equiv const.
\end{aligned}$$

First-order well-balanced scheme for the SWLME

Using the mid-point rule in (23) the first step is to obtain, if possible, the stationary solution W_i^* such that:

$$W_i^*(x_i) = W_i. \tag{30}$$

With this information the constants C_1, C_2, C_3,...,C_{N+2} can be computed as

$$\begin{cases} C_1 = h_i u_{m,i}, \\ C_2 = \frac{1}{2} u_{m,i}^2 + g(h_i + b(x_i)) + \frac{3}{2} \sum_{j=1}^N \frac{1}{2j+1} \alpha_{j,i}^2, \\ C_3 = \frac{\alpha_{1,i}}{h_i}, \\ C_4 = \frac{\alpha_{2,i}}{h_i}, \\ \quad \vdots \\ C_{N+2} = \frac{\alpha_{N,i}}{h_i}. \end{cases} \tag{31}$$

First-order well-balanced scheme for the SWLME

Using the relations (31), the stationary solution can be evaluated in a point $x = a$. The evaluation of the steady state solution requires finding roots of the function

$$f(h) = Dh^4 + 2h^3 g + 2h^2(gb(a) - C_2) + C_1^2, \qquad (32)$$

where the parameter D is given by

$$D = C_3^2 + \frac{3}{5}C_4^2 + \cdots + \frac{3}{2N+1}C_{N+2}^2.$$

First-order well-balanced scheme for the SWLME

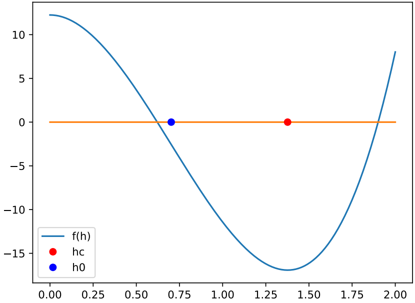


Figura: An example of the root finding function $f(h)$ (32) with some constants C_i. The minimum h_c and the initial value of the Newton algorithm h_0 are shown.

First-order well-balanced scheme for the SWLME

We can conclude then the following: If $f(h_c) < 0$ there exist two possible states for $W_i^*(x_{i\pm\frac{1}{2}})$, one subcritical and one supercritical. The following criterion will be used to choose one state:

1. If W_i is subcritical or supercritical, then we will choose the solution in the same regime (subcritical or supercritical) as W_i for $W_i^*(x_{i\pm\frac{1}{2}})$.
2. If W_i is transcritical, then the solution that has the same behaviour (subcritical or supercritical) as W_{i-1} will be selected for $W_i^*(x_{i-\frac{1}{2}})$ and the solution whose behaviour is the same as W_{i+1} will be selected for $W_i^*(x_{i+\frac{1}{2}})$.

First-order well-balanced scheme for the SWLME

Following the procedure described in (Castro & Parés, 2020), the reconstruction operator reduces to $\mathbb{P}_i(x) = W_i^*(x)$ and the first order numerical scheme reduces to:

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta r}(D_{i+\frac{1}{2}}^- + D_{i-\frac{1}{2}}^+), \tag{33}$$

for $W_{i-\frac{1}{2}}^+ = \mathbb{P}_i(x_{i-\frac{1}{2}})$ and $W_{i+\frac{1}{2}}^- = \mathbb{P}_i(x_{i+\frac{1}{2}})$, where we have used that $\mathbb{P}(x) = W_i^*(x)$ is a steady solution.
In the case we could not find such a stationary solution verifying (30) the standard trivial reconstruction is considered.

Second-order well-balanced scheme for the SWLME

The cell averages of the initial condition are again computed using the mid-point rule:

$$W_i^0 = W_0(x_i), \quad \text{for all } i.$$

1 Obtaining the steady solution: the steady state W_i^* needs to be found such that

$$W_i^*(x_i) = W_i. \tag{34}$$

We obtain the constants $C_1, C_2, C_3, \ldots, C_{N+2}$ as in (31), so the stationary solution can be evaluated in a point $x = a$.

2 Computing the fluctuations: $\{V_{i-1}, V_i, V_{i+1}\}$ in (24) are computed using the mid-point rule

$$\begin{aligned} V_{i-1} &= W_{i-1} - W_i^*(x_{i-1}), \\ V_i &= W_i - W_i^*(x_i) = 0, \\ V_{i+1} &= W_{i+1} - W_i^*(x_{i+1}). \end{aligned}$$

Second-order well-balanced scheme for the SWLME

3. Applying the reconstruction operator: After the fluctuations are computed, the *minmod* reconstruction is used to obtain the reconstruction operator

$$Q_i(x) = V_i + minmod\left(\frac{V_i - V_{i-1}}{\Delta x}, \frac{V_{i+1} - V_{i-1}}{2\Delta x}, \frac{V_{i+1} - V_i}{\Delta x}\right)(x - x_i),$$

where

$$minmod(a,b,c) = \begin{cases} min\{a,b,c\} & \text{if } \ a,b,c > 0, \\ max\{a,b,c\} & \text{if } \ a,b,c < 0, \\ 0 & \text{otherwise.} \end{cases}$$

4. Defining the well-balanced operator: The well-balanced reconstruction operator is given by

$$\mathbb{P}_i(x) = W_i^*(x) + Q_i(x).$$

Second-order well-balanced scheme for the SWLME

Replacing this in (29), we obtain:

$$
\begin{aligned}
W_i'(t) = & -\frac{1}{\Delta x}\Bigg(D^-_{i+\frac{1}{2}} + D^+_{i-\frac{1}{2}} \\
& + F(\mathbb{P}_i(x_{i+\frac{1}{2}})) - F(U_i^*(x_{i+\frac{1}{2}})) + F(U_i^*(x_{i-\frac{1}{2}})) - F(\mathbb{P}_i(x_{i-\frac{1}{2}})) \qquad (35) \\
& + B(\mathbb{P}_i(x_i)) minmod\left(\frac{V_i - V_{i-1}}{\Delta x}, \frac{V_{i+1} - V_{i-1}}{2\Delta x}, \frac{V_{i+1} - V_i}{\Delta x}\right)\Bigg).
\end{aligned}
$$

The discretization in time is performed with a Runge-Kutta TVD method of order 2.

Third-order well-balanced scheme for the SWLME

A third order well-balanced scheme will be based on the two point Gaussian quadrature formula for computing the averages. In the first step, we need to find the constants C_j, $j = 1, ..., N+2$ such that

$$\frac{1}{2} W_i^*(x_a, C_1, ..., C_{N+2}) + \frac{1}{2} W_i^*(x_b, C_1, ..., C_{N+2}) = W_i,$$

where x_a and x_b are the two quadrature points and $W_i^*(x, C_1, ..., C_{N+2})$ represents the stationary solution given by the constants C_j evaluated in x. Then we follow the steps considering a third order reconstruction operator (e.g. CWENO reconstruction) and using again the two point Gaussian quadrature.

Spatial discretization

Need to define the fluctuations $D^{\pm}_{i+\frac{1}{2}}$ and the non-conservative terms in (18), for which we use a path-conservative scheme based on segments in the conservative variables as family of paths joining two states:

$$\Psi(s; W_l, W_r) = \left(\begin{array}{c} \Psi_U(s; W_l, W_r) \\ \Psi_b(s; W_l, W_r) \end{array} \right) = \left(\begin{array}{c} U_l + s(U_r - U_l) \\ b_l + s(b_r - b_l) \end{array} \right), \ \ s \in [0,1],$$

and a PVM-like method (Castro & Fernández-Nieto,2012) corresponding to a choice in (20) of

$$D^{\pm}_{i+\frac{1}{2}} = \frac{1}{2}\Big(F(U_r) - F(U_l) + B_{i+\frac{1}{2}}(U_r - U_l) - S_{i+\frac{1}{2}}(b_r - b_l) \\ \pm Q_{i+\frac{1}{2}}(U_r - U_l - A^{-1}_{i+\frac{1}{2}} S_{i+\frac{1}{2}}(b_r - b_l))\Big), \tag{36}$$

where

$$\mathcal{A}_{i+\frac{1}{2}} = \left(\begin{array}{cc} A_{i+\frac{1}{2}} & -S_{i+\frac{1}{2}} \\ 0 & 0 \end{array} \right)$$

Spatial discretization

$A_{i+\frac{1}{2}} = J_{i+\frac{1}{2}} + B_{i+\frac{1}{2}}$ and $S_{i+\frac{1}{2}}$ have to verify

$$J_{i+\frac{1}{2}}(U_r - U_l) = F(U_r) - F(U_l), \tag{37}$$

$$B_{i+\frac{1}{2}} = \int_0^1 B(U_l + s(U_r - U_l))\, ds, \tag{38}$$

$$S_{i+\frac{1}{2}} = \int_0^1 S(U_l + s(U_r - U_l))\, ds, \tag{39}$$

and the polynomial viscosity matrix is $Q_{i+\frac{1}{2}} = P(A_{i+\frac{1}{2}})$, for polynomial P.

Spatial discretization

$$S_{i+\frac{1}{2}} = \begin{pmatrix} 0 \\ -g\frac{h_l+h_r}{2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

$$J_{i+\frac{1}{2}} = \frac{\partial F}{\partial U}(h_R, u_{m,R}, \alpha_{1,R}, ..., \alpha_{N,R}), \tag{40}$$

at the intermediate values

$$h_R = \frac{h_l + h_r}{2}, \quad u_{m,R} = \frac{\sqrt{h_l}u_{m,l} + \sqrt{h_r}u_{m,r}}{\sqrt{h_l} + \sqrt{h_r}},$$

and

$$\alpha_{j,R} = \frac{\sqrt{h_l}h_r\alpha_{j,r} + \sqrt{h_r}h_l\alpha_{j,l}}{\sqrt{h_l}h_r + \sqrt{h_r}h_l}, \quad j = \{1, ..., N\}.$$

Spatial discretization

$$B_{i+\frac{1}{2}} = diag(0, 0, -u_{m,b}, ..., -u_{m,b}), \tag{41}$$

at values

$$u_{m,b} = \begin{cases} \frac{h_r^2 u_r + h_l^2 u_l + h_l h_r \left[(u_l - u_r) log\left(\frac{h_r}{h_l}\right) - (u_r + u_l) \right]}{(h_r - h_l)^2} & \text{if } h_r \neq h_l, \\ \frac{u_r + u_l}{2} & \text{if } h_r = h_l. \end{cases}$$

We take an HLL-like method that correspond to an election of $P(x) = a_0 + a_1 x$ in (36). The coefficients are given as

$$a_0 = \frac{S_r|S_l| - S_l|S_r|}{S_r - S_l}, \quad a_1 = \frac{|S_r| - |S_l|}{S_r - S_l},$$

where S_r and S_l are the maximum and the minimum eigenvalue of $A_{i+\frac{1}{2}}$, respectively.

Numerical tests

- Well-balanced property:
  - Test 1: Lake at rest.
  - Test 2: Subcritical stationary solution.
  - Test 3: Transcritical stationary solution. Test 4: Subcritical stationary solution with non zero moments.
- Comparison between the SWLME, HSWME and βHSWME:
  - Test 5: Transient model comparison with standard dam-break test.
  - Test 6: Transient model comparison with square root velocity profile.

In all test cases we will use 1000-point uniform mesh, free boundary conditions, a CFL number of 0.5 and $N = 8$.
The first four test cases: $g = 9.812$.
The two last test cases: $g = 1$ and a flat bottom topography.

Test 1: Lake at rest

$$b_0(x) = \begin{cases} 2 - x^2 & \text{if } \; -0.5 < x < 0.5, \\ 1.75 & \text{otherwise,} \end{cases} \tag{42}$$

and therefore

$$W_0(x) = (h_0(x), u_{m,0}(x)h_0(x), \alpha_{1,0}(x)h_0(x), ..., \alpha_{N,0}(x)h_0(x)) = (3 - b_0(x), 0, 0, ..., 0). \tag{43}$$

Scheme (1000 cells)	$\|\|\Delta h\|\|_1$ (1st)	$\|\|\Delta u\|\|_1$ (1st)	$\|\|\Delta h\|\|_1$ (2nd)	$\|\|\Delta u\|\|_1$ (2nd)
Well-balanced	0.00	8.16e-16	0.00	8.16e-16
Non well-balanced	0.00	7.12e-16	4.51e-15	1.75e-14

Cuadro: Well-balanced vs non well-balanced schemes: L^1 errors $||\Delta \cdot ||_1$ at time $t = 0.5$ for the SWLME model with initial conditions (42) and (43).

Test 1: Lake at rest

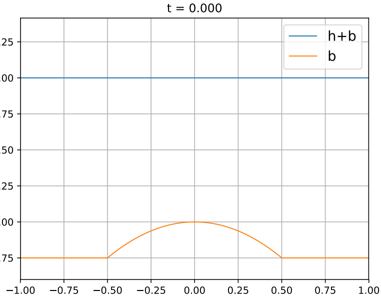


Figura: Initial condition for the lake at rest (42) and (43).

Test 2: Subcritical stationary solution

We take this test from (Castro, López-García & Parés, 2013). The bottom topography is chosen as

$$b_0(x) = \begin{cases} 0.25(1 + cos(5\pi(x+0.5))) & \text{if } 1.3 < x < 1.7, \\ 0 & \text{otherwise.} \end{cases} \quad (44)$$

As $W_0(x)$ we take the subcritical stationary solution such that $C_1 = 3.5$, $C_2 = 17.56957396120237$ and $C_i = 0$ for $i \in \{3, ..., N+2\}$

Scheme (1000 cells)	$\|\|\Delta h\|\|_1$ (1st)	$\|\|\Delta u\|\|_1$ (1st)	$\|\|\Delta h\|\|_1$ (2nd)	$\|\|\Delta u\|\|_1$ (2nd)
Well-balanced	9.16e-16	1.79e-15	1.42e-15	3.24e-15
Non well-balanced	2.48e-6	5.08e-6	3.21e-5	8.40e-5

Cuadro: Well-balanced vs non well-balanced schemes: L^1 errors $||\Delta \cdot ||_1$ at time $t = 0.5$ for the SWLME model with initial condition (44).

Test 2: Subcritical stationary solution.

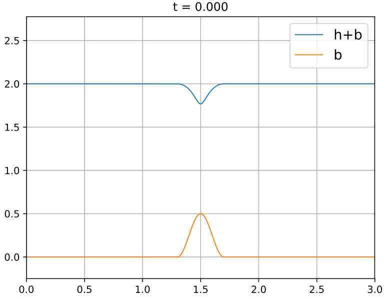


Figura: Initial condition for the subcritical stationary solution (44) for variable h

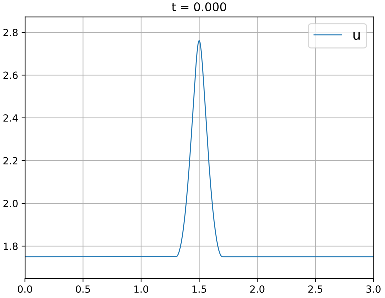


Figura: Initial condition for the subcritical stationary solution (44) for variable u

Test 3: Transcritical stationary solution

We take this test from (Castro, López-García & Parés, 2013). The bottom topography is chosen as

$$b_0(x) = \begin{cases} 0.25(1 + cos(5\pi(x + 0.5))) & \text{if } 1.3 < x < 1.7, \\ 0 & \text{otherwise.} \end{cases} \quad (45)$$

As $W_0(x)$ we take the transcritical stationary solution

$$W_0(x) = \begin{cases} W_*(x) & \text{if } x < 1.5 \\ W^*(x) & \text{if } x > 1.5 \end{cases} \quad (46)$$

where W_* and W^* are the subcritical and supercritical stationary solutions such that $C_1 = 2.5$, $C_2 = 21, 15525$ and $C_i = 0$ for $i \in \{3, ..., N + 2\}$.

Scheme (1000 cells)	$\|\|\Delta h\|\|_1$ (1st)	$\|\|\Delta u\|\|_1$ (1st)	$\|\|\Delta h\|\|_1$ (2nd)	$\|\|\Delta u\|\|_1$ (2nd)
Well-balanced	3.53e-14	2.95e-13	3.53e-14	2.98e-13
Non well-balanced	1.46e-5	1.22e-4	3.07e-4	1.12e-3

Cuadro: Well-balanced vs non well-balanced schemes: L^1 errors $||\Delta \cdot ||_1$ at time $t = 0.5$ for the SWLME model with initial condition (45) and (46).

Test 3: Transcritical stationary solution

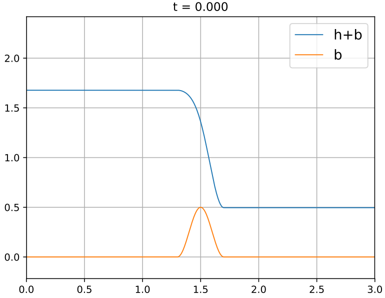


Figura: Initial condition for the subcritical stationary solution (45) and (46) for variable h

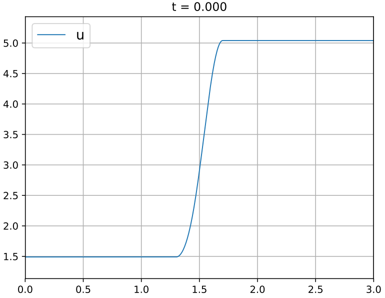


Figura: Initial condition for the subcritical stationary solution (45) and (46) for variable u

Test 4: Subcritical stationary solution with non zero moments

The bottom topography is chosen as

$$b_0(x) = \begin{cases} 0.25(1 + cos(5\pi(x + 0.5))) & \text{if } 1.3 < x < 1.7 \\ 0 & \text{otherwise} \end{cases} \tag{47}$$

As $W_0(x)$ we use the subcritical stationary solution such that $C_1 = 3.5$, $C_2 = 21,15525$ and $C_i = 0.25$ for $i \in \{3, \ldots, N+2\}$.

Scheme	$\|\|\Delta h\|\|_1$, 1st	$\|\|\Delta u\|\|_1$ (1st)	$\|\|\Delta\alpha_i\|\|_1$ (1st)	$\|\|\Delta h\|\|_1$ (2nd)	$\|\|\Delta u\|\|_1$ (2nd)	$\|\|\Delta\alpha_i\|\|_1$ (2nd)
wb	4.00e-15	9.71e-15	4.45e-15	2.56e-15	7.66e-15	5.04e-15
Non wb	3.11e-6	6.65e-6	6.98e-7	3.98e-5	1.04e-4	2.52e-5

Cuadro: Well-balanced (WB) vs non well-balanced schemes: L^1 errors $||\Delta \cdot ||_1$ at time $t = 0.5$ for the SWLME model with initial condition (47).

Test 4: Subcritical stationary solution with non zero moments

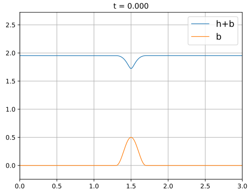


Figura: Initial condition for the subcritical stationary solution (47) for variable *h*

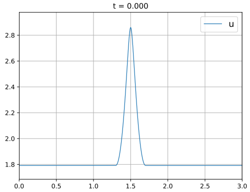


Figura: Initial condition for the subcritical stationary solution (47) for variable *u*

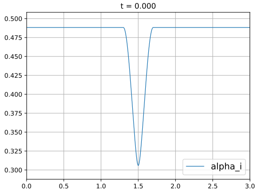


Figura: Initial condition for the subcritical stationary solution (47) for variable α_i

Test 5: transient model comparison with standard dam-break test

We are going to consider the following dam-break initial condition taken from (Koellermeier & Rominger, 2020) without friction terms

$$W_0(x) = (h_0(x), u_{m,0}(x)h_0(x), \alpha_{1,0}(x)h_0(x), ..., \alpha_{N,0}(x)h_0(x)), \tag{48}$$

where $u_{m,0}(x) = 0.25$, $\alpha_{1,0}(x) = -0.25$, $\alpha_{N,0}(x) = 0.25$, $\alpha_{i,0}(x) = 0, i \in \{2, ..., N-1\}$, and

$$h_0(x) = \begin{cases} 5 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases} \tag{49}$$

Test 5: transient model comparison with standard dam-break test

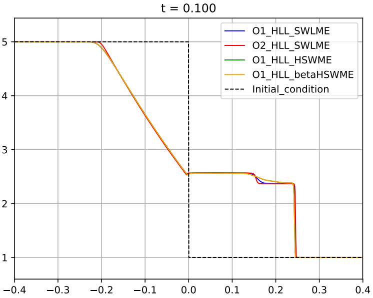


Figura: Water height *h*.

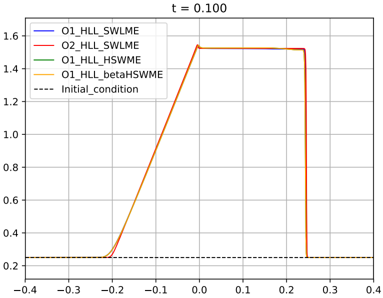


Figura: Velocity *u*.

Test 5: transient model comparison with standard dam-break test

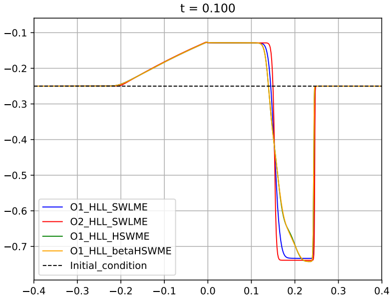


Figura: First coefficient α_1.

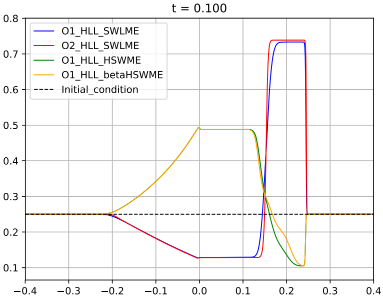


Figura: Last coefficient α_8.

Test 6: transient model comparison with square root velocity profile

For the last test, we consider the following dam-break initial condition:

$$W_0(x) = (h_0(x), u_{m,0}(x)h_0(x), \alpha_{1,0}(x)h_0(x), ..., \alpha_{N,0}(x)h_0(x)), \tag{50}$$

where we use a square root initial velocity profile (2) $u(0, x, \zeta) = u_m(0, x) + \sum_{j=1}^{N} \alpha_j(0, x)\phi_j(\zeta) = \sqrt{\zeta}$, such that the initial variables can be computed as $u_{m,0}(x) = 1$ and

$$\begin{aligned} \alpha_{1,0}(x) = -\tfrac{3}{5}, \quad \alpha_{2,0}(x) = -\tfrac{1}{7}, \quad \alpha_{3,0}(x) = -\tfrac{1}{15}, \quad \alpha_{4,0}(x) = -\tfrac{3}{77}, \\ \alpha_{5,0}(x) = -\tfrac{1}{39}, \quad \alpha_{6,0}(x) = -\tfrac{1}{55}, \quad \alpha_{7,0}(x) = -\tfrac{3}{221}, \quad \alpha_{8,0}(x) = -\tfrac{1}{95}. \end{aligned} \tag{51}$$

The initial water height is chosen as

$$h_0(x) = \begin{cases} 5 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases} \tag{52}$$

Test 6: transient model comparison with square root velocity profile

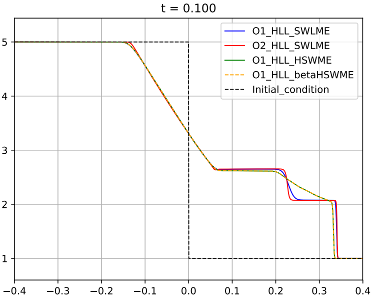


Figura: Water height h.

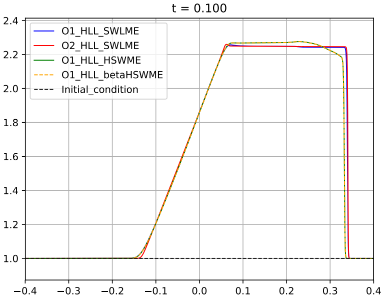


Figura: Velocity u.

Frame Title

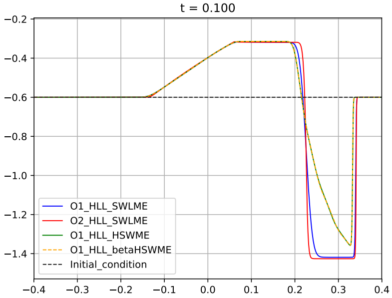


Figura: First coefficient α_1.

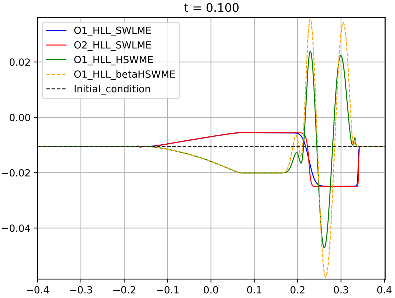


Figura: Last coefficient α_8.

Conclusions

- We derive a new model called Shallow Water Linearized Moment Equations (SWLME), based on a linearization during the derivation.
- The concise derivation of the SWLME allowed to prove hyperbolicity and to fully characterize its eigenstructure analytically.
- This information was used to define a first order and a second order well-balanced numerical scheme preserving the steady states of the model numerically up to machine precision.
- We compared the new SWLME model to other existing shallow water moment models, obtaining very similar solutions for the standard dam-break test.
- The solution for a more complex velocity profile seems more stable with the new SWLME model while existing models show emerging instabilities.

End

Thank you for your attention